Math 250 Briggs 4.8

Newton's Method

1) Use Newton's Method to find approximate values of the zeros of a function.

2) Understand why Newton's Method may sometimes fail for one starting value but succeed for a different starting value

3) Understand why Newton's Method sometimes fails for any starting value, and demonstrate this using algebra.

Newton's Method always used to solve stuff = 0.
Sometimes we arrange to get 0 on RHS.

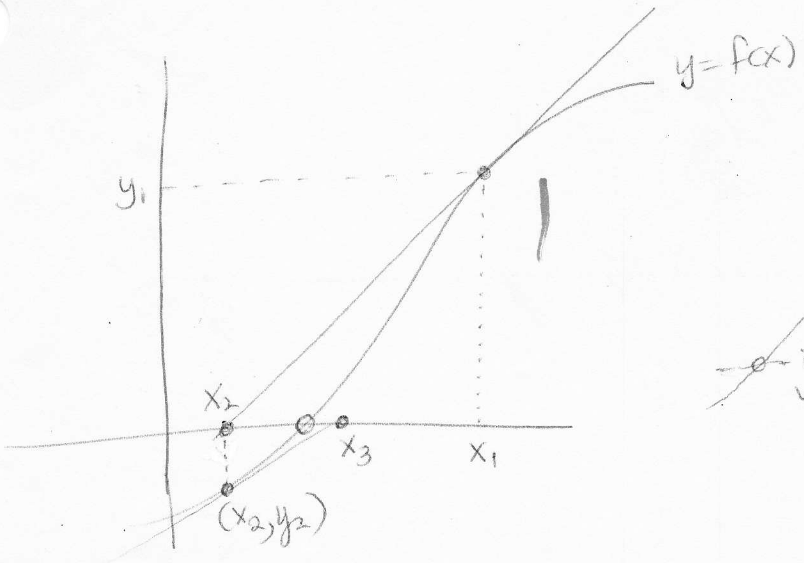
- Points of intersection $f(x)=g(x)$ means $\underbrace{f(x)-g(x)}_{\text{stuff}}=0$

- Fixed points $f(x)=x$ means $\underbrace{f(x)-x}_{\text{stuff}}=0$

- Local extrema (CV, actually) means $\underbrace{f'(x)}_{\text{stuff}}=0$

Math 250 Newton's Method.

Approximate a zero of function using Newton's method.



is the value of x we want to find

Write the equation of the line tangent to f(x) at (x_1, y_1).

$$y - y_1 = f'(x_1)(x - x_1)$$

Find the x-intercept of this line. [set $y=0$, solve for x.]

$$0 - y_1 = f'(x_1)(x - x_1)$$

$$\frac{-y_1}{f'(x_1)} = x - x_1$$

$$x_1 - \frac{y_1}{f'(x_1)} = x$$

$$x = x_1 - \frac{y_1}{f'(x_1)}$$

How was y_1 originally calculated? $y_1 = f(x_1)$.

Substitute that:

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Call this x-intercept $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Repeat the process. to get x_3

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

Repeat again to get $x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$.

* Memorize *

$$\boxed{\text{Newton's Method} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Math 250

If we are "lucky", the sequence

$x_1, x_2, x_3, x_4 \ldots$

will get closer and closer to the zero of the function $f(x)$.
This is called "converging", or a "convergent sequence".
If so, Newton's method succeeds.

The success or failure of Newton's Method can depend on several things.

Temporary: choose a new x_1

1) Initial guess x_1 may be too far from or too close to the zero.
2) Initial guess may have $f'(x_1) = 0$, causing a divide-by-zero error
3) A later value x_i may have $f'(x_i) = 0$, causing a divide-by-zero error.
4) Initial guess may have $f(x_1)$ or $f'(x_1)$ undefined.
5) A later value x_i may have $f(x_i)$ or $f'(x_i)$ undefined.

Permanent: do algebra to simplify $x - \dfrac{f(x)}{f'(x)}$

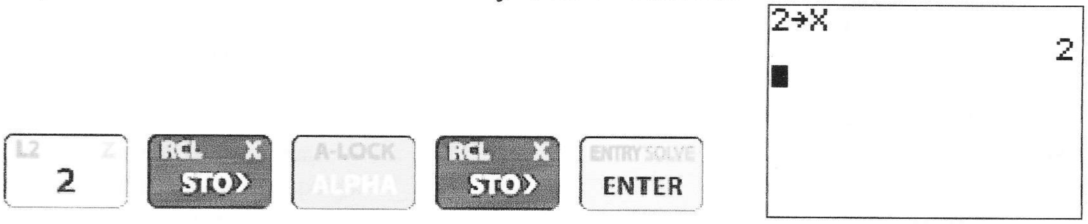
6) The fraction $\dfrac{f(x)}{f'(x)}$ simplifies to a constant so that $x + \dfrac{f(x)}{f'(x)}$ keeps getting bigger no matter what x_1 is chosen.

Note: If $f(x_1) = 0$, then your initial guess is already a solution of $f(x) = 0$, (an x-intercept or zero).

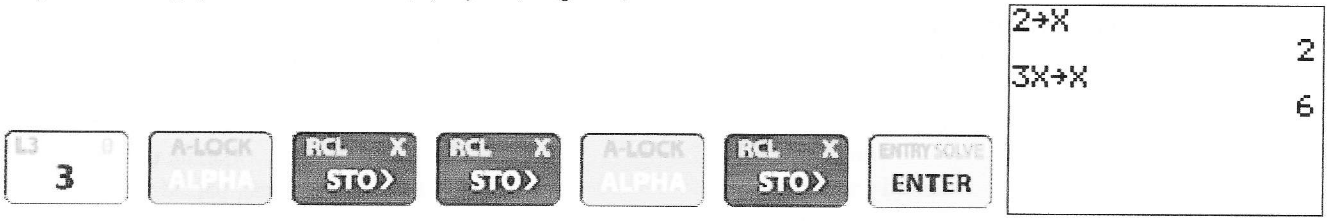
Math 250 Newton's Method Graphing Calculator Skills

<u>Skill I</u>. Calculate something with variable x and store the calculated result in x, replacing the original value of x

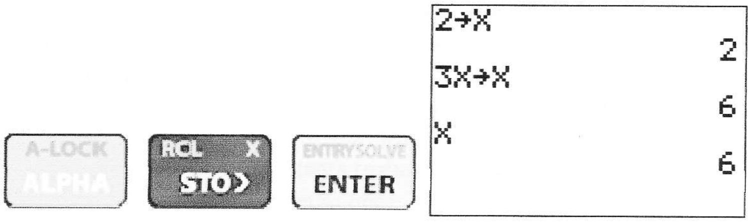
Step 1: Store the value 2 in the memory location called x.



Step 2: Multiply the value in x (2) by 3 (to get 6) and store the result in x.

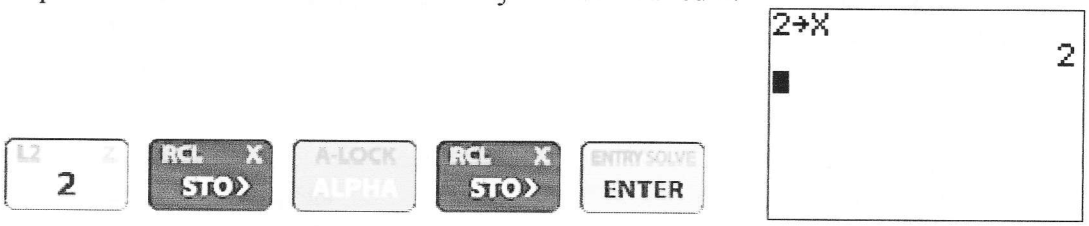


Step 3: Confirm that the value in x is 6.

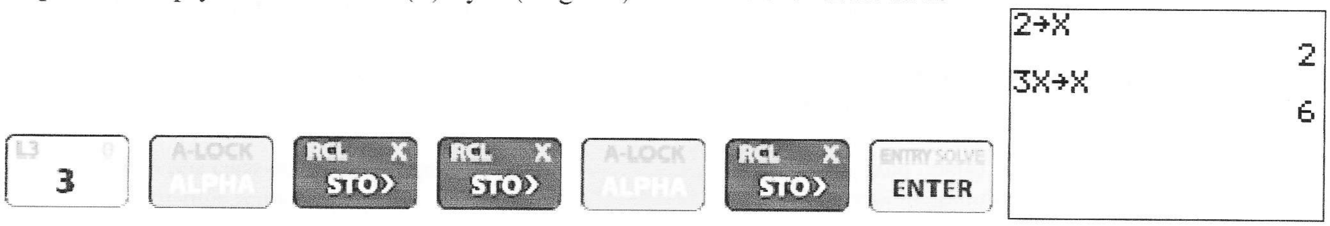


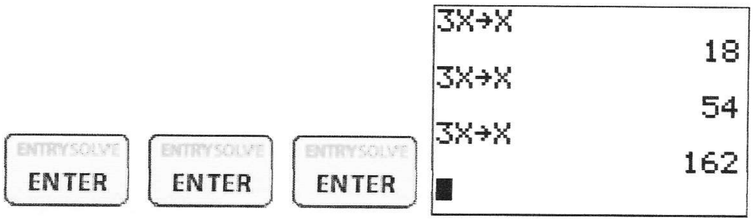
<u>Skill II</u>. Do Skill I repeatedly.

Step 1: Store the value 2 in the memory location called x.



Step 2: Multiply the value in x (2) by 3 (to get 6) and store the result in x.

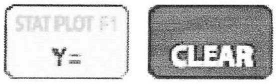




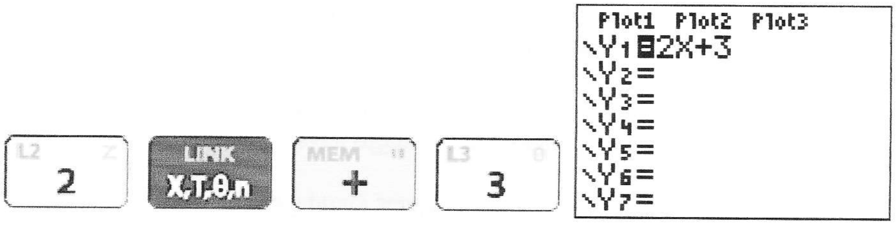
Step 3: Repeat the calculation by pressing Enter.
Each time we press Enter, the previous result is multiplied by 3 and stored in the same location.

Skill III. Evaluate a function value using the Y-VARS menu and memory storage locations.

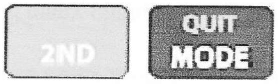
Step 0: Clear all functions from the Y= menu.



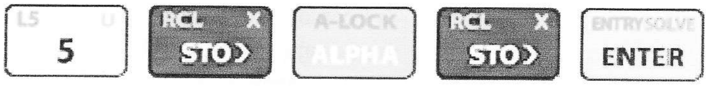
Step 1: Type in the function $y = 2x + 3$.



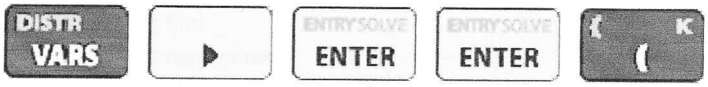
Step 2: Exit the Y= menu and return to the calculating screen.

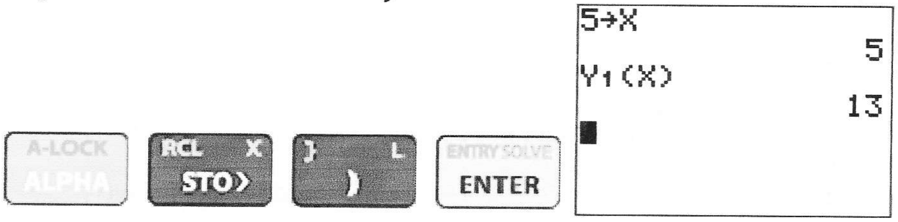


Step 3: Store the value 5 in the memory location x.



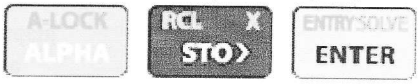
Step 4: Evaluate the function $y = 2x + 3$ for $x = 5$.

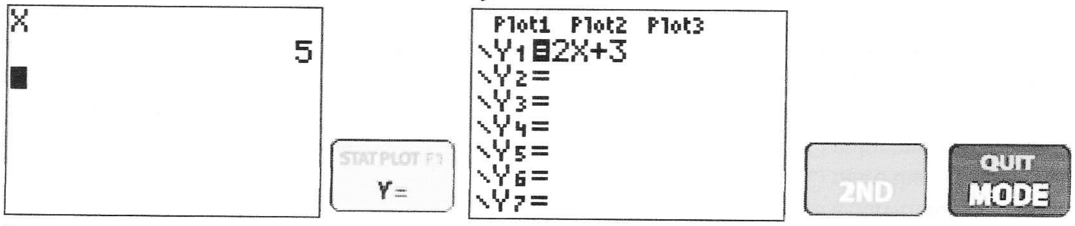




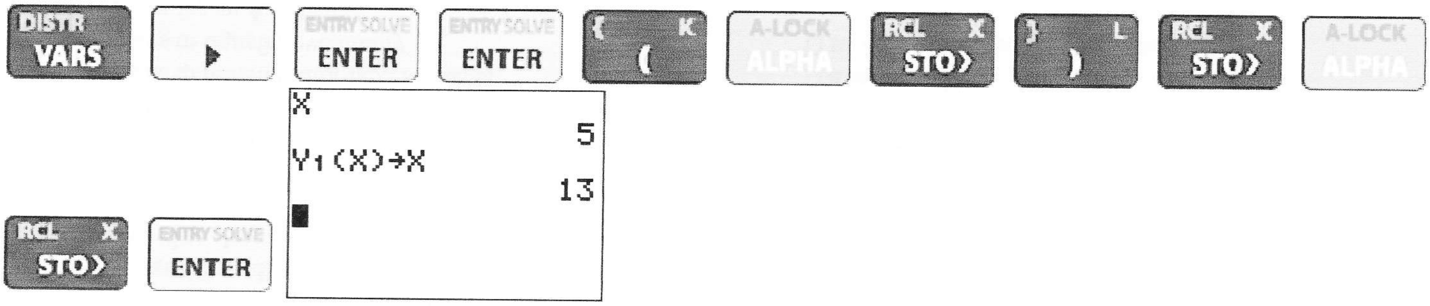
Skill IV. Repeatedly evaluate a function using value stored in x, and store the result in x. (x? Yes, x.)

Confirm that 5 is still stored in x and $y = 2x + 3$ is still in the Y= menu.

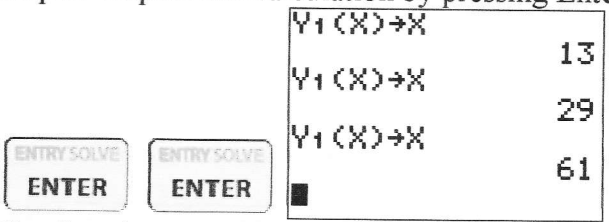




Step 1: Evaluate y(x) for x=5 and store the result in x.



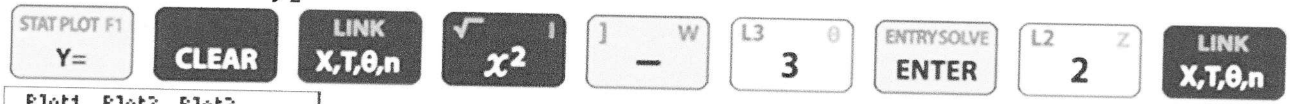
Step 2: Repeat this calculation by pressing Enter.

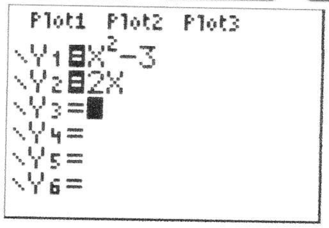


The first time, we find y(13)=2(13)+3=29. The next it's y(29)=2(29)+3=61.

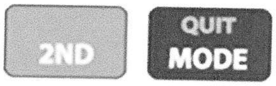
Skill V. Use the GC to calculate seven iterations of Newton's Method to find an approximate zero of $f(x) = x^2 - 3$. Use $x = 2$ as the initial guess.

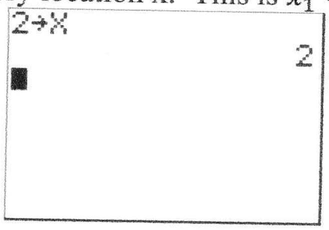
Step 1: Find $f'(x)$ analytically and input $f(x)$ and $f'(x)$ in the Y= menu. Hint: Always put the function in y_1 and the derivative in y_2.

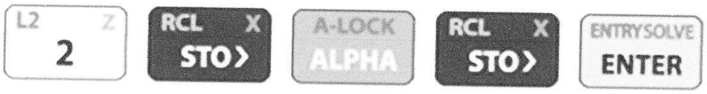




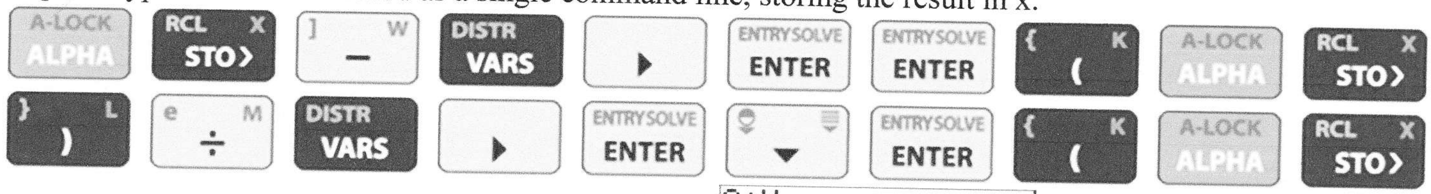
Step 2: Exit to calculating window and, store 2 in memory location x. This is $x_1 = 2$.

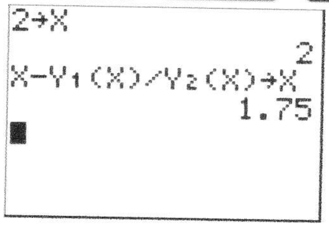


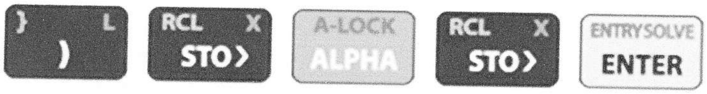




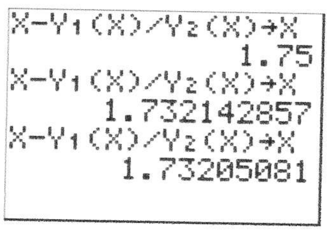
Step 3: Type Newton's Method as a single command line, storing the result in x.

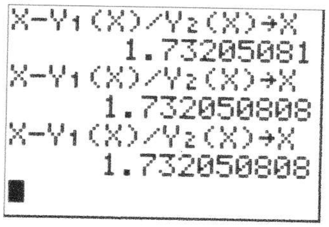






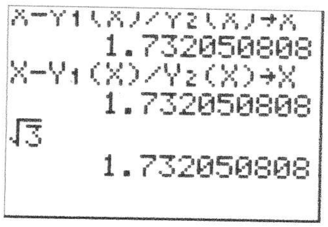
This is $x_2 = 1.75$



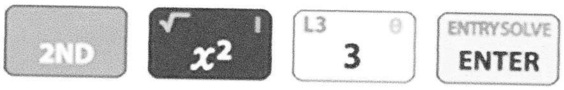


Step 4: Repeat this calculation by pressing Enter 5 times.

These are $x_3 = 1.732142857$, $x_4 = 1.73205081$, $x_5 = 1.73205081$, and $x_6 = x_7 = 1.732050808$. We have reached the accuracy of this calculator, and all future values will be the same.



Check: $\sqrt{3} = 1.732050808$



Note: $f(x) = x^2 - 3$ has a second zero at $x = -\sqrt{3}$. Use a different guess!

M250 Newton's Method

You may write tables of all the calculations:

$n \quad x_n \quad f(x_n) \quad f'(x_n) \quad \frac{f(x_n)}{f'(x_n)} \quad x_n - \frac{f(x_n)}{f'(x)}$ BUT....

Because we have an efficient way to do the entire calculation all at once, you do NOT need to write down such a table.
Instead you should show the following in your work:

$f(x)$

$f'(x)$.

The rule for using Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

All of your calculated sequence, rounded to at least two decimal places more than the prescribed tolerance.

(2) Use Newton's Method to calculate the zero of $f(x) = \tan(x)$ with initial guess $x_1 = 2.5$, until successive iterations differ by less than .0001.

$y_1 = f(x) = \tan(x)$

$y_2 = f'(x) = \sec^2 x = 1/(\cos(x))^2$

$x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \frac{\tan(x_n)}{\sec^2(x_n)}$

using $\boxed{\text{TABLE}}$ in $\boxed{\text{Ask}}$

Residual $f(x_i)$

$x_1 = 2.5$ — $\tan(2.5) \approx -0.747$

$x_2 \approx 2.979462$ — 6 places — $\tan(2.979462) \approx -0.1636$

$x_3 \approx 3.138766$ — $\tan(3.138766) \approx -0.0028$

$x_4 \approx 3.141593$ — $\tan(3.141593) \approx 3.5 \times 10^{-7}$

$x_5 \approx 3.141593$

$\boxed{\tan(3.14159) \approx 0.}$

OR zero $\boxed{x \approx 3.14159}$

(3) What happens if you use $x_1 = \pi/2$? $\Longrightarrow$ $\pi/2$ not in domain of tan.
AND $\cos(\pi/2) = 0$ div by 0.

So Newton's Method can fail if the initial guess is chosen poorly.

(4) Approximate the zero(s) of $f(x) = \sqrt[5]{x}$ using Newton's method.

$f(x) = x^{1/5}$

$f'(x) = \frac{1}{5} x^{-4/5}$

$x_{n+1} = x_n - \frac{x_n^{1/5}}{\frac{1}{5} x_n^{-4/5}} = x_n - 5x_n = -4x_n$

m250

ex (4) cont.

whatever your initial guess, the next iteration is 4 times the magnitude and the opposite sign.

This is an example of a function where Newton's method fails for <u>every</u> initial guess.

This is because $f'(x) = \frac{1}{5}x^{-4/5}$ is not defined (vertical tangent) at the zero $x=0$.

<u>Summary of Newton's Method</u>

- Used to find values of x, call it $x=c$, where $f(c)=0$. {based on x-ints, it only works for x-ints, also called zeros}.
- the function f must be differentiable on some open interval containing c.
- Select an initial guess that $f'(x_1) \neq 0$ {no divide by 0 happens} and x_1 is in domain of f.

If Newton's method does not converge, try:

1. Simplify the expression to see if it's not supposed to converge.
2. Choose a new x_1.
3. Check that you correctly calculated $f'(x)$.
4. Check that you correctly typed $y_1 = f(x)$ and $y_2 = f'(x)$ into your calculator.
5. Check that you typed Newton's method into the calculator correctly.

{eg. $x_n + \dfrac{f(x_n)}{f'(x_n)}$ will foul up!}

{eg. $x_n - \dfrac{f'(x_n)}{f(x_n)}$ will too}

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(5) Apply Newton's Method to approximate the x-values of the point(s) of intersection of the two graphs until successive iterations differ by 0.001.

$f(x) = 3 - x$

$g(x) = \dfrac{1}{x^2+1}$

*
To use Newton's for Intersections
$Y_1 = f(x) - g(x)$
$Y_2 = f'(x) - g'(x)$

Newton's method is for <u>zeros</u>.

$$\boxed{\begin{array}{l}\text{If } f(x) = g(x) \text{ at point of intersection, then} \\ f(x) - g(x) = 0 \text{ at point of intersection}\end{array}}$$

Use Newton's method on $h(x) = 3 - x - \dfrac{1}{x^2+1}$

$$h'(x) = -1 - \frac{\left[(x^2+1)\cdot 0 - 1(2x)\right]}{(x^2+1)^2}$$

$$h'(x) = -1 + \frac{2x}{(x^2+1)^2}$$

Newton's method $x_{n+1} = x_n - \dfrac{\left[3 - x_n - \dfrac{1}{x_n^2+1}\right]}{\left[-1 + \dfrac{2x_n}{(x_n+1)^2}\right]}$

zero between $x = 2$ and $x = 3$?

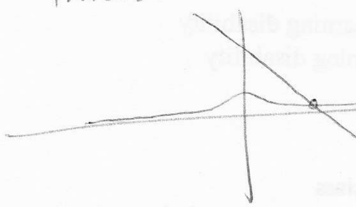
$x_1 = 3$

$x_2 \approx 2.89362$

$x_3 \approx 2.89329$

$\boxed{x \approx 2.893}$

Looks like only one point of intersection



(6) Approximate the fixed point of $f(x) = \cot x$ $0 < x < \pi$ to 2 decimal places. A fixed point x_0 is one where $f(x_0) = x_0$.

Use $g(x) = f(x) - x$

when $f(x) = x$

then $f(x) - x = 0$ need zero.

$g(x) = \cot(x) - x$

$g'(x) = -\csc^2(x) - 1$

$$x_{n+1} = x_n - \frac{\cot(x_n) - x_n}{-\csc^2(x_n) - 1}$$

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⑥ cont.

$y_1 = \cos(x)/\sin(x) - x$

$y_2 = -1/(\sin(x))^2 - 1$

$x_1 = 1$

$x_2 \approx .85163$

$x_3 \approx .86029$

$x_4 \approx .86033$

$x_5 \approx .86033\ldots$

Residual $\cot(x_i) - x_i = g(x_i)$

$g(1) \approx -.3579$

$g(.85163) \approx 0.02396$

$g(.86029) \approx 1.2 \times 10^{-4}$

$g(.86033) \approx 9.8 \times 10^{-6}$

$\boxed{x \approx .86}$ is approximate fixed pt. $\cot(.86) \approx .86$.

Convergence of Newton's Method

If $\left| \dfrac{f(x) \cdot f''(x)}{[f'(x)]^2} \right| < 1$ for all $x \in (a,b)$ where zero $c \in (a,b)$

then Newton's Method will converge.

This is called a "sufficient" condition because if it happens, its sufficient to ensure convergence.

Unfortunately, it's not a "necessary" condition, meaning that there are situations which converge but don't meet this condition.

Math 250

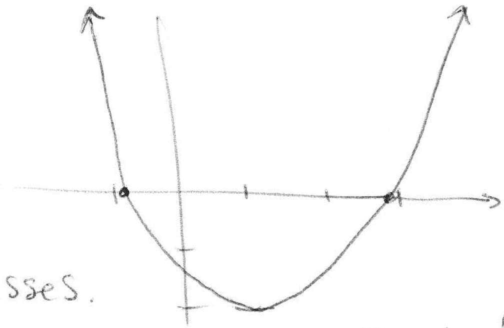
(7) Find all zeros (to four places) of $f(x) = x^2 - 2x - 2$ using Newton's Method.

$Y_1 = f(x) = x^2 - 2x - 2$

$Y_2 = f'(x) = 2x - 2$

check graph of $f(x)$ has 2 x-intercepts.

will need 2 initial guesses.



	Residual		Residual
$x_1 = 4$	6	$x_1 = -1$	1
$x_2 = 3$	1	$x_2 = -0.75$	.0625
$x_3 = 2.75$	.0625	$x_3 = -0.73214$	3.1×10^{-4}
$x_4 = 2.73214$	3.1×10^{-4}	$x_4 = -0.73205$	-3×10^{-6}
$x_5 = 2.73205$	-3×10^{-6}	$x_5 = -0.73205$	
$x_6 = 2.73205$			

$\boxed{x \approx 2.7321}$ $\boxed{x \approx -0.7321}$

Math 250 4.8 Briggs

(8) Estimate the critical values of $f(x) = x^5 - x^4 + x^2 - x$ on $(-\infty, \infty)$ using Newton's method.

C.V. means $f'(x) = 0$ or $f'(x)$ undefined (none for polynomial)

$f'(x) = 5x^4 - 4x^3 + 2x - 1 = 0$

means we need

$f''(x) = 20x^3 - 12x^2 + 2$

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$ when we apply Newton's to $f'(x)$.

$x_0 = 1$	$x_0 = -1$
$x_1 = .8$	$x_1 = -.8$
$x_2 = .6684$	$x_2 = -0.7060$
$x_3 = .6147$	$x_3 = -0.6844$
$x_4 = .6080$	$x_4 = -0.6834$
$\boxed{x_5 = .6080}$	$\boxed{x_5 = -0.6833}$